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Simultaneous Approximation of Vector-valued Functions

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INTRODUCTION

Approximation problems involving composite norms of vector-valued functions have been analyzed by several authors (see, e.g., Laurent [7, 8] and Bredendiek [2, 3]). These contributions mainly consist of ad hoc analyses of given settings where particular composite norms are to be minimized. In this paper, we embed the general problem of simultaneous approximation in an appropriate product space, constructed from a finite number of normed linear spaces. Once formulated this way, functional analysis provides us with the relevant information. The major advantage of this natural embedding is to produce a global theory for simultaneous approximation. The contribution of this paper consequently consists of obtaining a explicit formulation of the general characterization conditions in vectorial approximation. Clearly, these results enable further refinements for these characterizations.

In the first section we state the simultaneous approximation problem for which characterizations are to be found, and we recall briefly the optimality conditions of general approximation theory. In the second section we collect pertinent facts from functional analysis on the dual space of the normed product space concerning the explicit form of both the linear functionals involved and the extreme points of the dual unit ball. This enables us in the third section to obtain the searched for characterizations easily. Finally, they are applied to some examples.

1. FORMULATION OF THE PROBLEM

Throughout this paper we consider a finite family of normed linear spaces over the real or complex field. The cartesian product of these spaces will be denoted as $E = \prod_{i=1}^{n} E_i$. For any n-tuple $x \in E$ we denote by $x_i = \mathrm{Pr}_i(x)$

the projection onto E_i, and by $\| x_i \|_i$ the associated norm. The product space E will be provided with a norm $\| \cdot \|$, to be specified later from the norms given in the spaces E_i. In general, it is not necessary to state explicitly the relations between the components of the vectors in E. However, for elements of $M \subset E$, we assume here that the components depend on a parameter, an element of a linear space F. In other words we define $M = \{(g_1(a),..., g_n(a)) \mid a \in A\}$, where A is a subset of F, and each g_i is a mapping of F into E_i. An alternative presentation, often used in literature on vector-valued approximation, considers the elements of E to have components which are the images by n operators P_1,..., P_n of a given function, an element of a normed linear space E'. Consequently, $(P_1 f,..., P_n f)$ in $\prod_{i=1}^n E_i$ corresponds to $f \in E'$.

The general approximation problem by elements of a subset $M \subset E$, consists of determining best approximations from M to a given element $f \in E$ or at least of obtaining appropriate characterizations which may lead to a constructive scheme. Best approximations to f form a subset of M, which is the image of f through the set valued mapping, called metric projection $P_M: E \to \mathscr{P}(M)$, defined for any $f \in E$ as

$$P_M(f) = \{x \in M \mid \| f - x \| = d(f, M)\}, \tag{1}$$

where $d(f, \cdot)$ denotes the distance functional of f associated with the norm over E. To avoid trivial problems, we require that M is not dense in E, and restrict P_M to the subset $E \backslash \mathrm{cl}\, M$. Indeed on M the metric projection reduces to the canonical injection of M into E.

Approximation theory provides us with characterization conditions in the form of assertions on the existence of particular linear functionals, elements of the conjugate space $E^* = \mathscr{L}(\prod_{i=1}^n E_i, \mathbb{R})$ which is a Banach space for the usual norm $\| L \| = \sup\{| L(x)| \mid \| x \| \leqslant 1\}$. This conjugate space is further provided with the weak* topology $\sigma(E^*, E)$. A particular subset of the dual unit ball $B(E^*)$ is the set $\mathscr{M}_x = \{L \in E^* \mid \| L \| \leqslant 1, L(x) = \| x \|\}$ which is nonempty if $x \in E \backslash \{\theta\}$, and is extremal. According to [5], we also introduce the cone $\mathbb{C}[m, M]$ of adherent displacements of M starting from $m \in M$, which is the set of elements h of E, such that for any strictly positive scalar ϵ, and in any neighborhood N_h of h, there exists an element $h' \in N_h$ and a scalar $\eta \in]0, \epsilon[$ such that $m + \eta \cdot h'$ is in M. The largest linear subspace of E over $\mathbb{R}$, contained in the convex cone $\mathbb{C}[m, M]$ is given by $(-\mathbb{C}[m, M] \cap \mathbb{C}[m, M])$. The following characterization condition is known.

LEMMA 1 [5, Theorem 8]. *Let M be a subset of the normed linear space E, $f \in E \backslash \mathrm{cl}\, M$ and $m_0 \in M$. Also let S be a nonvoid linear subspace of E contained in the cone $\mathbb{C}[m_0, M]$, and $S^\perp$ the annihilator of S in E^*.*

(a) *If $m_0 \in P_M(f)$, then the set*

$$\mathscr{M}_{f-m_0} \cap S^{\perp} \tag{2}$$

is nonvoid in E^*.

(b) *If the Local Kolmogoroff condition on M versus S is sufficient, then $m_0 \in P_M(f)$ if and only if (2) is a nonempty set in E^*.*

We recall that the Local Kolmogoroff condition on M versus S is an always necessary condition, which can be stated as: If $m_0 \in P_M(f)$, then $\min\{\operatorname{Re} L(h) \mid L \in \mathscr{E}(\mathscr{M}_{f-m_0})\} \leqslant 0$, h independent in S. This minimum is taken over all extreme points of the set $\mathscr{M}_{f-m_0}$.

For the particular case M is a convex set ($M = C$), a general characterization condition can be formulated, similar to Lemma 1, where $S^{\perp}$ in (2) is to be replaced by a subset which is the normal cone $N[m_0, C]$ to C at m_0. The latter set is known to be the polar set of the cone $\mathbb{C}[m_0, C]$ [6, p. 24]. We have

LEMMA 2 [6, p. 76]. *Let C be a convex subset of the normed linear space E, $f \in E \backslash \operatorname{cl} C$ and $m_0 \in C$. The following statements are equivalent.*

(a) $m_0 \in P_C(f)$.

(b) *The set*

$$\mathscr{M}_{f-m_0} \cap N[m_0, C] \tag{3}$$

is a nonempty subset of E^*.

Finally, if C is a linear variety, $C = \omega_0 + V$ (V a linear subspace of E), we see that Lemma 2 is nothing else than the classical characterization theorem of linear approximation theory, since $N[m_0, C] = V^{\perp}$ in (3).

The above characterizations are stated in their most general form. Indeed until now, we did not use the fact that E is a product space.

2. PRELIMINARIES

We are interested in those norms over E, which can be described in terms of the norms associated with the spaces E_i. We shall restrict ourselves to the particular norms

$$\| x \|_{l\infty} = \max\{\| x_i \|_i \mid i = 1,..., n\}, \tag{4a}$$

$$\| x \|_{lp} = \left(\sum_{i=1}^{n} (\| x_i \|_i)^p \right)^{1/p}, \qquad \text{for} \quad 1 \leqslant p < \infty, \tag{4b}$$

E being then respectively denoted by $E_{l\infty}$ and E_{lp}. It is clear that the norms $\|\cdot\|_{l1}$, $\|\cdot\|_{l2}$, and $\|\cdot\|_{l\infty}$ are equivalent, since the corresponding distances are equivalent. Indeed the following inequality is valid for any $x \in E$: $\|x\|_{l\infty} \leqslant \|x\|_{l2} \leqslant \|x\|_{l1} \leqslant n\|x\|_{l\infty}$. Consequently, these different norms all define the same topology on E which is the so-called product topology. Moreover, by resorting to Jensen's theorem we have that $\|x\|_{l\infty} \leqslant \|x\|_{lm} \leqslant \|x\|_{lp}$ for any $x \in E$ and $1 \leqslant p < m < \infty$. Clearly, this proves that all norms (4) are equivalent, and all purely topological properties, unlike the metric ones, remain unchanged for any of these norms.

An equally fundamental remark holds for the connection between E^* and the spaces $E_i^* = \mathscr{L}(E_i, \mathbb{R})$. By [4, pp. 33–36] we have that the space $(\prod_{i=1}^n E_i^*)_{l1}$ is isometric to $(E_{l\infty})^*$, via the mapping T: $\prod_{i=1}^n E_i^* \to (\prod_{i=1}^n E_i)^*$, such that $T(L_1, ..., L_n) = L$, and for any $x \in E_{l\infty}$, we have that $L(x) = \sum_{i=1}^n L_i(x_i)$ and $\|L\| = \|(L_1, ..., L_n)\|_{l1}$. Two normed linear spaces E and F are isometric if there exists a linear mapping T of E onto F such that for any $x \in E$, $\|Tx\|_F = \|x\|_E$ [4, p. 30]. By this definition, T is also injective since $Tx = 0$ implies $x = 0$. Hence the mapping T^{-1} exists and both T and T^{-1} are linear, continuous one-to-one mappings. Similarly, we also have that $(\prod_{i=1}^n E_i^*)_{lp}$ is isometric to the space $(E_{lq})^*$, where $p^{-1} + q^{-1} = 1$ with $1 < p < \infty$, and $(\prod_{i=1}^n E_i^*)_{l\infty}$ is isometric to $(E_{l1})^*$, both under the same sort of mappings. If we replace a given norm in E by an equivalent one, then the norm on the product of the conjugate spaces E_i^*: $i = 1, ..., n$ is also replaced by an equivalent one. Moreover, for the element $(L_1, ..., L_n) \in \prod_{i=1}^n E_i^*$, we have that $\|(L_1, ..., L_n)\|_{l\infty} \leqslant \|(L_1, ..., L_n)\|_{lm} \leqslant \|(L_1, ..., L_n)\|_{lp}$ is valid, where $1 \leqslant p < m < \infty$. We obtain the following inclusion for the dual unit balls.

$$B\left(\prod_{i=1}^n E_i^*\right)_{lp} \subset B\left(\prod_{i=1}^n E_i^*\right)_{lm} \subset B\left(\prod_{i=1}^n E_i^*\right)_{l\infty} = \prod_{i=1}^n B(E_i^*). \tag{5}$$

By the Alaoglu theorem, every unit ball $B(E_i^*)$ is $\sigma(E_i^*, E_i)$-compact. Since the weak* topology $\sigma(\prod_{i=1}^n E_i^*, \prod_{i=1}^n E_i)$ is the product topology of the separate weak* topologies $\sigma(E_i^*, E_i)$ [1, p. 55], all balls $B(\prod_{i=1}^n E_i^*)_{lp}$, $1 \leqslant p \leqslant \infty$, of (5) are compact for $\sigma(\prod_{i=1}^n E_i^*, \prod_{i=1}^n E_i)$. By these remarks we can describe the extremal subset of $B(E^*)$, namely $\mathscr{M}_x$, taking into account that E is a product space. We obtain

$$\mathscr{M}_x = \left\{T(L_1, ..., L_n) \mid (L_1, ..., L_n) \in \prod_{i=1}^n E_i^*, \|(L_1, ..., L_n)\|_{l\gamma} = 1, \right.$$
$$\left. \sum_{i=1}^n L_i(x_i) = \|x\|_{l\beta}\right\}, \tag{6}$$

where $\gamma = \beta/\beta - 1$ if $1 < \beta < \infty$ and $\gamma = 1$ (resp. $=\infty$) if $\beta = \infty$ (resp. $=1$). It is interesting to remark here that relation (6) can be broken down into three specific, but more refined, ones. Considering first E_{l1}, then we have for any $x \in E_{l1}\backslash\{\theta\}$ that the statement $T(L_1, ..., L_n) \in \mathscr{M}_x$ is equivalent with

$$L_i \in B(E_i^*), \qquad L_i(x_i) = \| x_i \|_i \qquad \text{for} \quad i = 1,..., n. \tag{6a}$$

Indeed, if $T(L_1, ..., L_n) \in \mathscr{M}_x$, we have that $\| L_i \| \leqslant 1$, $\forall i = 1,..., n$ and $\sum_{i=1}^n (\operatorname{Re} L_i(x_i) - \| x_i \|_i) = 0$ together with $\operatorname{Re} L_i(x_i) \leqslant \| x_i \|_i$. Consequently $\operatorname{Re} L_i(x_i) = \| x_i \|_i$ and $L_i(x_i)$ is real and positive, for all $i = 1,..., n$. In case all $\| x_i \|_i \neq 0$, (6a) is equivalent with the requirement $L_i \in \mathscr{M}_{x_i}$, $i = 1,..., n$. For the space E_{lp} with $1 < p < \infty$, we have that

$$\mathscr{M}_x = \left\{ T(L_1, ..., L_n) \mid (L_1, ..., L_n) \in \prod_{i=1}^n E_i^*, \right.$$
$$\left. \sum_{i=1}^n L_i(x_i) = \left(\sum_{i=1}^n \| x_i \|_i^p \right)^{1/p}, \sum_{i=1}^n \| L_i \|^q = 1 \right\}, \tag{6b}$$

where $q = p/p - 1$. Finally, for $E_{l\infty}$, the statement $T(L_1, ..., L_n) \in \mathscr{M}_x$, is equivalent to: There exists a nonvoid subset $I \subset \{1,..., n\}$ such that

$$\begin{aligned} &\sum_{i \in I} \| L_i \| = 1, \qquad L_i(x_i) = \| L_i \| \cdot \| x \|_{l\infty} && \text{for} \quad i \in I, \\ &L_i = \theta && \text{for} \quad i \in \{1,..., n\}\backslash I. \end{aligned} \tag{6c}$$

This can readily be verified. Indeed if $T(L_1, ..., L_n) \in \mathscr{M}_x$, then we have that $\sum_{i=1}^n \| L_i \| = 1$ and $\sum_{i=1}^n L_i(x_i) = \| x \|_{l\infty} \geqslant \| x_i \|_i$, for $i = 1,..., n$. Denoting I the subset of $\{1,..., n\}$ such that $L_i \neq \theta$ and also $L_i' = L_i/\| L_i \|$ we have $\sum_{i=1}^n \| L_i \| \cdot (\operatorname{Re} L_i'(x_i) - \| x_i \|_i) = 0$. Consequently, $\operatorname{Re} L_i'(x_i) = L_i'(x_i) = \| x_i \|_i = \| x \|_{l\infty}$, for all $i \in I$.

Finally, it is useful to recall a result concerning the extremal points of the dual unit ball $B(E^*)$. By the mentioned isometry T between $\prod E_i^*$ and E^*, we have that T and T^{-1} are also linear continuous mappings between these locally convex topological linear spaces. Consequently we can apply a theorem (see [8, p. 436]). Since $B(\prod_{i=1}^n E_i^*)_{l\gamma}$ is a convex and compact set in the product topology of the $\sigma(E_i^*, E_i)$ and $T[B(\prod_{i=1}^n E_i^*)_{l\gamma}] = B(E_{l\beta}^*)$ we have that $\mathscr{E}(B(E_{l\beta}^*)) \subset T[\mathscr{E}(B(\prod_{i=1}^n E_i^*)_{l\gamma})]$. Similarly, for T^{-1} and by the fact that $B(E_{l\beta}^*)$ is a convex and $\sigma(E^*, E)$-compact set, we have $\mathscr{E}(B(\prod_{i=1}^n E_i^*)_{l\gamma}) \subset T^{-1}[\mathscr{E}(B(E_{l\beta}^*))]$. The ensuing identity is then

$$\mathscr{E}(B(E_{l\beta}^*)) = T\left[\mathscr{E}\left(B\left(\prod_{i=1}^n E_i^*\right)_{l\gamma}\right)\right]. \tag{7}$$

For some particular norms in $(\prod_{i=1}^{n} E_i^*)$, the extremal points of the dual unit ball can be described more explicitly. Indeed, it is easily verified that the extremal points of the unit balls $B(\prod_{i=1}^{n} E_i^*)_{l1}$ and $B(\prod_{i=1}^{n} E_i^*)_{l\infty}$ are, respectively, given as

$$\mathscr{E}\left(B\left(\prod_{i=1}^{n} E_i^*\right)_{l1}\right) = \Big\{(L_1, \ldots, L_n) \in \prod_{i=1}^{n} E_i^* \mid \exists\omega \in \{1,\ldots, n\};$$

$$L_\omega \in \mathscr{E}(B(E_\omega^*)),\ L_i = \theta,\ \forall i = 1,\ldots, n,\ i \neq \omega\Big\}, \quad \text{(7a)}$$

and

$$\mathscr{E}\left(B\left(\prod_{i=1}^{n} E_i^*\right)_{l\alpha}\right) = \Big\{(L_1, \ldots, L_n) \in \prod_{i=1}^{n} E_i^* \mid L_i \in \mathscr{E}(B(E_i^*)),\ \forall i \in \{1,\ldots, n\}\Big\}. \quad \text{(7b)}$$

3. Characterization Conditions

These results play an important role in obtaining optimality conditions for the particular problem of simultaneous approximation of vector-valued functions. Taking into account relation (6), Lemma 1 can immediately be restated in the following form.

THEOREM 3. *Let $M = \{(\prod_{i=1}^{n} g_i(a)) \mid a \in A\}$ be a subset of the normed linear space $E = (\prod_{i=1}^{n} E_i)_{l\beta}$ where $1 \leqslant \beta \leqslant \infty$, $f \in E \backslash \mathrm{cl}\, M$, and $g(a_0) = \prod_{i=1}^{n} g_i(a_0) \in M$. Let S be a nonempty linear subspace of E, contained in the cone $\mathbb{C}[g(a_0), M]$.*

(a) *If $g(a_0) \in P_M(f)$, then there exist linear functionals $(L_1, \ldots, L_n) \in \prod_{i=1}^{n} E_i^*$ such that*

(1) $\|(L_1, \ldots, L_n)\|_{l\gamma} = 1$, *where $\gamma = \beta/(\beta - 1)$ if $1 < \beta < \infty$, $\gamma = 1$ if $\beta = \infty$, and $\gamma = \infty$ if $\beta = 1$;*

$$\begin{aligned} &(2)\quad \sum_{i=1}^{n} L_i(h_i) = 0 \qquad (h_1, \ldots, h_n) \in S; \\ &(3)\quad \sum_{i=1}^{n} L_i(f_i - g_i(a_0)) = \|f - g(a_0)\|_{l\beta}. \end{aligned} \quad (8)$$

(b) *If the Local Kolmogoroff condition on M versus S is sufficient then we have that $g(a_0) \in P_M(f)$, if and only if there exist linear functionals $(L_1, \ldots, L_n) \in \prod_{i=1}^{n} E_i^*$ such that (8) is satisfied.*

If A is a subset of the Banach space F, and the mapping $g: A \to \prod_{i=1}^{n} E_i$ is Gateaux (resp. Frechet) differentiable, provided that the Gateaux (resp. Frechet) derivative at $a_0 \in \operatorname{int} A$ exists, then the linear subspace S can be taken as $\{[Dg(a_0)] \cdot b \mid b \in F\}$, where $[Dg(a_0)] \in \mathscr{L}(F, E)$ is the Gateaux (resp. Frechet) derivative [5].

Similarly, if the set M is a convex set C, we obtain by Lemma 2

THEOREM 4. *Let $C = \{\prod_{i=1}^{n} g_i(a) \mid a \in A\}$ be a convex set of the normed linear space $E = (\prod_{i=1}^{n} E_i)_{l\beta}$ where $1 \leqslant \beta \leqslant \infty$, $f \in E \backslash \operatorname{cl} C$, and $g(a_0) = \prod_{i=1}^{n} g_i(a_0) \in C$. The following statements are equivalent.*

(a) $g(a_0) \in P_C(f)$.

(b) *There exist linear functionals $(L_1, ..., L_n) \in \prod_{i=1}^{n} E_i^*$ such that* (8.1), (8.3), *and*

$$\operatorname{Re}\left(\sum_{i=1}^{n} L_i\, g_i(a_0)\right) = \max\left\{\operatorname{Re} \sum_{i=1}^{n} L_i g_i(a) \mid g(a) \in C\right\} \tag{9}$$

are satisfied.

More particularly, if C is a linear variety, $C = \omega_0 + V$, we have that (8) is a necessary and sufficient condition for $g(a_0)$ to be best approximation of f in this linear variety, taking $S = V$. Especially if V is finite-dimensional, we have Theorem 4, where C is then a translated m-dimensional linear subspace V and $g(a_0) = \omega_0 + (\sum_{j=1}^{m} a_{0j}\Phi_{1j}, ..., \sum_{j=1}^{m} a_{0j}\Phi_{nj}) \in C = \omega_0 + V$. Statement (9) becomes

$$\sum_{i=1}^{n} L_i(\Phi_{ij}) = 0 \qquad j = 1, ..., m. \tag{10}$$

Statements (8.1) and (8.3) in the Theorems 3 and 4 can further be particularized taking into account (6a and c), depending on the value of β. We have then for $\beta = 1$ that (8.1) and (8.3) are equivalent to

(8.1′) $L_i \in B(E_i^*)$ $\qquad i = 1, ..., n$;

(8.3′) $L_i(f_i - g_i(a_0)) = \|f_i - g_i(a_0)\|_i$ $\qquad i = 1, ..., n$;

and for $\beta = \infty$, that (8.1) and (8.3) are equivalent to: There exists a subset I of $\{1, ..., n\}$ such that

(8.1″) $\sum_{i \in I} \| L_i \| = 1, \qquad L_i = \theta \qquad$ for $\quad i \in \{1, ..., n\} \backslash I$;

(8.3″) $L_i(f_i - g_i(a_0)) = \| L_i \| \cdot \|f - g(a_0)\|_{l\infty} \qquad i \in I.$

To obtain further characterizations of practical value, we now resort to some refinements. According to [5, Lemma 15], we have in the particular case that the linear subspace S is d-dimensional, that whenever the linear functional L is an element of the set described in (2), there equivalently exist h linear functionals $\{\mathscr{L}_1, ..., \mathscr{L}_h\} \subset \mathscr{E}(B(E^*))$ and h strictly positive scalars $\rho_1, ..., \rho_h$, where $1 \leqslant h \leqslant d+1$ for a real E and $1 \leqslant h \leqslant 2d+1$ for a complex E, such that $\sum_{j=1}^{h} \rho_j = 1$, $\sum_{j=1}^{h} \rho_j \mathscr{L}_j(h) = 0$ for all $h \in S$, and $\sum_{j=1}^{h} \rho_j \mathscr{L}_j(f - m_0) = \|f - m_0\|$. The latter equality is equivalent with the requirement $\mathscr{L}_j(f - m_0) = \|f - m_0\|$ for $j = 1, ..., h$. To apply this decomposition, in the context of product spaces, we only need to focus our attention on the statement $\mathscr{L}_j \in \mathscr{E}(B(E^*))$. By (7) we have that the existence of a linear functional ($\mathscr{L}_j$) extremal point of the dual unit ball $B(E_{l\beta}^*)$ is equivalent with the existence of $(L_{j1}, ..., L_{jn}) \in \mathscr{E}(B(\prod_{i=1}^{n} E_i^*)_{l_\gamma})$ such that $T(L_{j1}, ..., L_{jn}) = \mathscr{L}_j$. Consequently, from Lemma 1, we obtain the following main characterization theorem, for the problem under investigation.

THEOREM 5. *Let $M = \{\prod_{i=1}^{n} g_i(a) \mid a \in A\}$ be a subset of the normed linear space $E = (\prod_{i=1}^{n} E_i)_{l\beta}$, where $1 \leqslant \beta \leqslant \infty$, $f \in E \backslash \mathrm{cl}\, M$, and $g(a_0) = \prod_{i=1}^{n} g_i(a_0) \in M$. Let S be a nonempty finite-dimensional linear subspace of E, contained in $\mathbb{C}[g(a_0), M]$, $d = \dim S$.*

(a) *If $g(a_0) \in P_M(f)$*

(1) *then there exist linear functionals*

$$(L_{j1}, ..., L_{jn}) \in \mathscr{E}\left(B\left(\prod_{i=1}^{n} E_i^*\right)_{l_\gamma}\right) \qquad j = 1, ..., h; \tag{11.1}$$

(2) *there also exist h strictly positive scalars $\rho_1, ..., \rho_h$, where $1 \leqslant h \leqslant d+1$ for a real E and $1 \leqslant h \leqslant 2d+1$ for a complex E, such that*

$$\sum_{j=1}^{h} \rho_j \sum_{i=1}^{n} L_{ji}(h_i) = 0 \qquad (h_1, ..., h_n) \in S; \tag{11.2}$$

and

$$(3) \quad \sum_{i=1}^{n} L_{ji}(f_i - g_i(a_0)) = \|f - g(a_0)\|_{l\beta} \qquad j = 1, ..., h. \tag{11.3}$$

(b) *If the Local Kolmogoroff condition on M versus S is sufficient, then we have that $g(a_0) \in P_M(f)$, if and only if the conditions* (11.1, 2, *and* 3) *of part* a *are satisfied.*

Similar decompositions are possible for Lemma 2. In particular, if M is a linear variety, we obtain

COROLLARY 6. *If M is a translated d-dimensional linear subspace V of the normed linear space $E = (\prod_{i=1}^{n} E_i)_{l\beta}$, where $1 \leqslant \beta \leqslant \infty$, $f \in E \backslash M$, and $g(a_0) = \omega_0 + (\sum_{j=1}^{d} a_{0j}\Phi_{1j}, \ldots, \sum_{j=1}^{d} a_{0j}\Phi_{nj}) \in M = \omega_0 + V$, then the following statements are equivalent.*

(a) $g(a_0) \in P_M(f)$.

(b) *There exist linear functionals* (11.1) *and h strictly positive scalars $\rho_1, \ldots, \rho_h$, where $1 \leqslant h \leqslant d + 1$ for a real E and $1 \leqslant h \leqslant 2d + 1$ for a complex E, such that* (11.3) *and*

$$\sum_{j=1}^{h} \rho_j \sum_{i=1}^{n} L_{ji}(\Phi_{il}) = 0 \qquad l = 1, \ldots, d. \tag{12}$$

Further refinements of the foregoing decomposition are readily obtained if the product space E is endowed with the norm (4a) or the norm (4b), where $p = 1$. Indeed for $E = (\prod_{i=1}^{n} E_i)_{l1}$, by (7b) and (6a) we have that both the conditions (11.1) and (11.3) in Thoerem 5 and Corollary 6 become:

(11.1′) There exist h distinct sets of n linear functionals $(L_{j1}, \ldots, L_{jn})$ for $j = 1, \ldots, h$, and $L_{ji} \in \mathscr{E}(B(E_i^*))$ for $i = 1, \ldots, n$, where $1 \leqslant h \leqslant d + 1$ for a real E and $1 \leqslant h \leqslant 2d + 1$ for a complex E.

(11.3′) $L_{ji}(f_i - g_i(a_0)) = \|f_i - g_i(a_0)\|_i \quad \forall (i, j) \in \{1, \ldots, n\} \times \{1, \ldots, h\}$.

On the other hand, considering $E = (\prod_{i=1}^{n} E_i)_{l\infty}$ we have by (7a) and (6c) that (11.1 and 3) in Theorem 5 and Corollary 6 can be stated as:

(11.1″) For any $j = 1, \ldots, h$, there exists a unique index $I(j) \in \{1, \ldots, n\}$ such that $L_{jI(j)} \in \mathscr{E}(B(E_{I(j)}))$, and $L_{ji} = \theta$ for $i \in \{1, \ldots, n\} \backslash I(j)$, where $1 \leqslant h \leqslant d + 1$ for a real E and $1 \leqslant h \leqslant 2d + 1$ for a complex E.

(11.3″) $L_{jI(j)}(f_{I(j)} - g_{I(j)}(a_0)) = \|f - g(a_0)\|_{l\infty} \quad j = 1, \ldots, h$.

Clearly by (11.3″) we have that $\|f_{I(j)} - g_{I(j)}(a_0)\|_{I(j)} = \|f - g(a_0)\|_{l\infty}$ holds for $j = 1, \ldots, h$.

4. SOME APPLICATIONS

By the information gathered in the preceding sections, most applications become straightforward. We consider a wider class of norms. We define

$$n(x) = \max\{\lambda_i \| x_i \|_i \mid i = 1, \ldots, n\}, \tag{13a}$$

and also

$$N(x) = \sum_{i=1}^{n} \lambda_i \| x_i \|_i , \tag{13b}$$

where all λ_i are arbitrary positive scalars. In order to obtain adequate characterizations for the approximation problem of $(f_1, \ldots, f_n)$ by elements of $M = \{\prod_{i=1}^n g_i(a) \mid a \in A\}$, we apply the preceding theory on the modified problem which consists of approximating the function $(\lambda_1 f_1, \ldots, \lambda_n f_n)$ by elements of $\{\prod_{i=1}^n \lambda_i\, g_i(a) \mid a \in A\}$ for the norm (4a) or (4b), where $p = 1$. In a first example we consider the product space $(C(Q))^n$ endowed with the norm (13a), where $\| x_i \|_i = \max\{| x_i(q)| \mid q \in Q\}$ is the norm of $x_i = \mathrm{Pr}_i(x) \in C(Q)$. The following characterization follows immediately from Corollary 6 and (11.1″ and 3″).

COROLLARY 7. *Let* $V = \{\prod_{i=1}^n (\sum_{k=1}^m a_k \Phi_{ik}) \mid (a_1, \ldots, a_m) \in \mathscr{C}^m\}$ *be an m-dimensional linear subspace of* $(C(Q))^n$, *endowed with* (13a), $v(\alpha) \in V$, *and* $f = (f_1, \ldots, f_n) \in (C(Q))^n \backslash V$. *The following statements are equivalent.*

(a) $v(\alpha) \in P_V(f)$.

(b) *There exist h points* $q_1, \ldots, q_h$ *in* Q *and h nonzero scalars* μ_j, $j = 1, \ldots, h$, *where* $1 \leqslant h \leqslant 2m + 1$, *and, with every* $j = 1, \ldots, h$, *there exists a unique index* $I(j) \in \{1, \ldots, n\}$ *such that*

$$\lambda_{I(j)}[f_{I(j)}(q_j) - v_{I(j)}(\alpha, q_j)] = \operatorname{sign} \mu_j \cdot n[f - v(\alpha)] \qquad j = 1, \ldots, h,$$

and

$$\sum_{j=1}^{h} \lambda_{I(j)} \cdot \mu_j \Phi_{I(j)l}(q_j) = 0 \qquad l = 1, \ldots, m.$$

In a second example, the product space $(C(Q))^n$ is endowed with the norm (13b). The following characterization can be obtained.

COROLLARY 8. *Let* $V = \{\prod_{i=1}^n (\sum_{k=1}^m a_k \Phi_{ik}) \mid (a_1, \ldots, a_m) \in \mathscr{C}^m\}$ *be an m-dimensional linear subspace of* $(C(Q))^n$, *endowed with* (13b), $v(\alpha) \in V$, *and* $f = (f_1, \ldots, f_n) \in (C(Q))^n \backslash V$. *The following statements are equivalent.*

(a) $v(\alpha) \in P_V(f)$.

(b) *There exist h distinct sets, each of n points,* $(x_1, y_1, \ldots, z_1), \ldots, (x_h, y_h, \ldots, z_h) \in Q^n$ *and h strictly positive scalars* ρ_j *where* $1 \leqslant h \leqslant 2m + 1$, *and there also exist* $(n \cdot h)$ *scalars* ϵ_{ji}, $\forall (j, i) \in \{1, \ldots, h\} \times \{1, \ldots, n\}$ *such that* $| \epsilon_{ji} | = 1$,

$$\sum_{j=1}^{h} \rho_j(\lambda_1 \epsilon_{j1} \Phi_{1k}(x_j) + \cdots + \lambda_n \epsilon_{jn} \Phi_{nk}(z_j)) = 0 \qquad k = 1, \ldots, m,$$

and

$$\epsilon_{j1}\left[f_1(x_j)-\sum_{k=1}^{m}\alpha_k\Phi_{1k}(x_j)\right]=\left\|f_1-\sum_{k=1}^{m}\alpha_k\Phi_{1k}\right\|_1 \qquad j=1,\ldots,h$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\epsilon_{jn}\left[f_n(z_j)-\sum_{k=1}^{m}\alpha_k\Phi_{nk}(z_j)\right]=\left\|f_n-\sum_{k=1}^{m}\alpha_k\Phi_{nk}\right\|_n \qquad j=1,\ldots,h.$$

It is interesting to observe the difference between the structure of the error curves in the two foregoing examples. In the first case there are h points in Q, which are distributed among the different subproblems. Each partial error curve attains the global norm at these points. There may be some partial curves where the global norm is not attained. In the second case, for every partial error curve associated with a subproblem there exist points (at least 1) where the partial norm is attained. But in most problems there are no points where the global norm is attained.

Finally in a third example, we consider the product space $E \times F$ endowed with the norm $N(x_1, x_2) = \lambda_1 \| x_1 \|_E + \lambda_2 \| x_2 \|_F$, where λ_1 and λ_2 are arbitrary positive scalars, $E = C(Q)$ with $\| x_1 \|_E = \max\{| x_1(q)| \mid q \in Q\}$, and $F = L_p(B)$, where $1 < p < \infty$, and $\| x_2 \|_F = (\int_B [x_2(t)]^p \, dt)^{1/p}$. The following characterization is easily deduced for the linear approximation problem.

COROLLARY 9. *Let* $V = \{(\sum_{k=1}^{m} a_k\Phi_k, \sum_{k=1}^{m} a_k\Psi_k) \mid (a_1, \ldots, a_m) \in \mathscr{C}^m\}$ *be an m-dimensional linear subspace of* $C(Q) \times L_p(B)$, *endowed with* $N(x_1, x_2)$, $v(\alpha) \in V$, *and* $f = (f_1, f_2) \in [C(Q) \times L_p(B)] \setminus V$. *The following statements are equivalent.*

(a) $v(\alpha) = (\sum_{k=1}^{m} \alpha_k\Phi_k, \sum_{k=1}^{m} \alpha_k\Psi_k) \in P_V(f)$.

(b) *There exist* h *points* $q_j \in Q$ *and* h *nonzero scalars* ρ_j, *where* $1 \leqslant h \leqslant 2m+1$, *such that* $\sum_{j=1}^{h} |\rho_j| = 1$,

$$\sum_{j=1}^{h}\rho_j\left[f_1(q_j)-\sum_{k=1}^{m}\alpha_k\Phi_k(q_j)\right]=\left\|f_1-\sum_{k=1}^{m}\alpha_k\Phi_k\right\|_E,$$

and

$$\lambda_1\sum_{j=1}^{h}\rho_j\Phi_k(q_j)+\lambda_2\int_B\Psi_k(t)\frac{|f_2(t)-\sum_{k=1}^{m}\alpha_k\Psi_k(t)|^{p-1}}{\|f_2(t)-\sum_{k=1}^{m}\alpha_k\Psi_k(t)\|_F^{p-1}}$$

$$\cdot\operatorname{sign}\left[f_2(t)-\sum_{k=1}^{m}\alpha_k\Psi_k(t)\right]dt=0, \qquad \textit{for} \quad k=1,\ldots,m.$$

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REFERENCES

1. N. BOURBAKI, "Espaces Vectoriels Topologiques," Chaps. 3–5, Hermann, Paris, 1955.
2. E. BREDENDIEK, Simultanapproximationen, *Arch. Rational Mech. Anal.* **33** (1969), 307–330.
3. E. BREDENDIEK, Charakterisierung und Eindeutigkeit bei Simultanapproximationen, *Z. Angew. Math. Mech.* **50** (1970), 403–410.
4. M. DAY, "Normed Linear Spaces," Springer-Verlag, Berlin, 1973.
5. C. DIERIECK, Characterizations for best nonlinear approximations, a geometrical interpretation, *J. Approximation Theory* **14** (1975), 163–187.
6. R. HOLMES, "A Course on Optimization and Best Approximation," Springer-Verlag, Berlin, 1972.
7. P. J. LAURENT, Théorèmes de caractérisation d'une meilleure approximation dans un espace normé et généralisation de l'Algorithme de Rémès, *Numer. Math.* **10** (1967), 190–208.
8. P. J. LAURENT, "Approximation et Optimisation," Hermann, Paris, 1972.